

# Time-varying bang-bang property of minimal controls for approximately null-controllable heat equations\*

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## Abstract

In this paper, optimal time control problems and optimal target control problems are studied for the approximately null-controllable heat equations. Compared with the existed results on these problems, the boundary of control variables are not constants but time varying functions. The time-varying bang-bang property for optimal time control problem, and an equivalence theorem for optimal control problem and optimal target problem are obtained.

**Keywords:** Heat equation, bang-bang property, optimal time control problem, optimal target control problem

**AMS subject classification:** 35K05, 49J20

## 1 Introduction

Let  $T$  be a positive number (can be taken  $\infty$ ) and  $\Omega$  be an open bounded domain with smooth boundary in  $\mathbb{R}^N$ ,  $N \geq 1$ . Let  $\omega$  be an open set of  $\Omega$ . Consider the following controlled system:

$$(1.1) \quad \begin{cases} \partial_t y(x, t) - \Delta y(x, t) = \chi_\omega \chi_{(\tau, T)} u(x, t), & \text{in } \Omega \times (0, T), \\ y(x, t) = 0, & \text{on } \partial\Omega \times (0, T), \\ y(x, 0) = y_0(x), & \text{in } \Omega. \end{cases}$$

Here  $y_0 \in L^2(\Omega)$  is a given initial data,  $u \in L^\infty(0, T; L^2(\Omega))$ ,  $\tau \in [0, T)$ ,  $\chi_\omega$  and  $\chi_{(\tau, T)}$  stand for the characteristic functions of  $\omega$  and  $(\tau, T)$ , respectively. We denote the solution to (1.1) by  $y(\cdot; \chi_{(\tau, T)} u, y_0)$  with initial data  $y_0$  and control  $u$ .

In this paper, denote by  $L_+^\infty(0, T)$  the subset of  $L^\infty(0, T)$ , whose element is almost surely positive. Denote by  $\|\cdot\|$  and  $\langle \cdot, \cdot \rangle$  the norm and inner product of  $L^2(\Omega)$ , respectively, and  $B(0, r)/\bar{B}(0, r)$  the open/closed ball of  $L^2(\Omega)$  with center 0 and radius  $r > 0$ .

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The null and approximate controllability of system (1.1) has been studied in many works (see, e.g. [2, 5]). Especially, for each  $\varepsilon > 0$ , since the system (1.1) is energy decaying, taking  $u = 0$ , we have  $\|y(t; 0, y_0)\| \leq \varepsilon$ , when  $t$  is large enough. By this, we can easily see that the system (1.1) is approximately null-controllable for large  $T$ . The reader can also refer to [1, 7, 8, 14, 15] for more discussions on controlled heat equations.

Three kinds of optimal control problems: the optimal time, target and norm control problems are important and interesting branches of optimization. For the deterministic systems, the reader can refer to [4] to obtain the recent results and open problems. The reader can also refer to [3, 9, 10, 11, 12, 14] for the optimal time control problems. For the stochastic ones, the optimal norm control problems were considered in [18, 19, 21] for controlled stochastic ordinary differential equations, and in [20] for controlled stochastic heat equations. The reader can also refer to [6, 15, 17] for the work on equivalence relation between these three optimal control problems.

For a given function  $M(\cdot) \in L_+^\infty(0, T)$ , we can define the admissible control set of controlled system (1.1):

$$\mathcal{U}_M \equiv \{v \in L^\infty(0, T; L^2(\Omega)) \mid \|v(t)\| \leq M(t) \text{ for a.e. } t \in (\tau, T)\},$$

and can denote the reachable set of system (1.1) with  $u \in \mathcal{U}_M$  by

$$\mathcal{R}(y_0, \tau, T) = \{y(T; \chi_{(\tau, T)} u, y_0) \mid u \in \mathcal{U}_M\}$$

and

$$\mathcal{R}(y_0, T) = \bigcup_{\tau \in (0, T)} \mathcal{R}(y_0, \tau, T).$$

By above discussion, without loss of generality, we can assume that  $\mathcal{R}(y_0, T) \cap \bar{B}(0, \varepsilon) \neq \emptyset$  for  $\varepsilon > 0$  and  $T > 0$ . We need note that  $y(T; \chi_{(\tau, T)} 0, y_0)$  may not be in  $\bar{B}(0, \varepsilon)$ .

Consider the following optimal time control problem

$$(1.2) \quad \tau(\varepsilon) = \sup_{\tau \in [0, T]} \{\tau \mid y(T; \chi_{(\tau, T)} u, y_0) \in \bar{B}(0, \varepsilon), u \in \mathcal{U}_M\}.$$

If the optimal time control problem (1.2) is solvable, i.e., there exist at least one  $u^* \in \mathcal{U}_M$  such that  $y(T; \chi_{(\tau(\varepsilon), T)} u^*, y_0) \in \bar{B}(0, \varepsilon)$ , we call that  $u^*$  an *optimal time control*. By choosing the minimal sequence and applying the classical variational method, we can prove that the optimal time control problem (1.2) has a solution  $u^*$  (see Lemma 2.1). What we are interested in is the following problem:

*Is the optimal time control  $u^*$  satisfying the time-varying bang-bang property?*

The (time-invariant) bang-bang property is a classical problem in control theory. There are many works on this topic (see, e.g. [12, 13, 14]). In [14], the author obtained the following the bang-bang property of a null-controlled heat equation. For a given positive constant  $M_0$ , define

$$\mathcal{U}_{M_0} = \{v \in L^\infty(0, T; L^2(\Omega)) \mid \|v(t)\| \leq M_0 \text{ for a.e. } t \in (\tau, T)\}.$$

Then, if  $u^* \in \mathcal{U}_{M_0}$  is an optima time control respect to (1.2), then the following time-invariant bang-bang property holds:

$$\|u^*(t)\| = M_0 \text{ for a.e. } t \in (\tau(0), T).$$

[16] owns many interesting results, but the bang-bang property is also depend on the positive constant  $M_0$ . This work is inspired by [14, 16]. In this work, we study the time-varying bang-bang property of an approximately null-controllable heat equation. Compared with the problem studied in [14], the boundary is not a constant  $M_0$  but a function  $M(\cdot)$ . Hence, the method used in [14] is not workable any more. Until now, to our best knowledge, there does not exist any work on this kind time-varying bang-bang property. The following is our first main result.

**Theorem 1.1.** *Suppose that  $M(\cdot) \in L_+^\infty(0, T)$ ,  $\varepsilon > 0$ , and  $\mathcal{R}(y_0, T) \cap \bar{B}(0, \varepsilon) \neq \emptyset$ . Then there exists a unique optimal time control  $u^* \in \mathcal{U}_M$  such that the optimal time control problem (1.2) is solvable. Moreover, the optimal control  $u^*$  satisfies the following time-varying bang-bang property:*

$$(1.3) \quad \|u^*(t)\| = M(t) \text{ for a.e. } t \in (\tau(\varepsilon), T).$$

Now, we consider the following optimal target control problem:

$$(1.4) \quad \varepsilon(\tau) = \inf \left\{ \|y(T; \chi_{(\tau, T)} u, y_0)\| \mid u \in \mathcal{U}_M \right\}.$$

Define

$$\varepsilon_T = \|y(T; \chi_{(0, T)} 0, y_0)\|.$$

It is obviously that  $0 \leq \varepsilon(\tau) \leq \varepsilon_T$  and  $0 \leq \tau(\varepsilon) \leq T$ . As an application of the time-varying bang-bang property, we shall give our second main result: a kind of equivalence related to  $\varepsilon(\tau)$  and  $\tau(\varepsilon)$ .

**Theorem 1.2.** *Let  $M(\cdot) \in L_+^\infty(0, T)$ . Then the map  $\tau \mapsto \varepsilon(\tau)$  is strictly monotonically increasing and continuous from  $[0, T)$  onto  $[\varepsilon(0), \varepsilon_T)$ . Furthermore, it holds that*

$$(1.5) \quad \varepsilon = \varepsilon(\tau(\varepsilon)), \varepsilon \in [\varepsilon(0), \varepsilon_T), \quad \text{and} \quad \tau = \tau(\varepsilon(\tau)), \tau \in [0, T).$$

Consequently, the maps  $\tau \mapsto \varepsilon(\tau)$  and  $\varepsilon \mapsto \tau(\varepsilon)$  are inverse of each other.

When  $M(\cdot) \equiv M_0$  ( $M_0 > 0$  is a constant), a kind of equivalence theorem of optimal time and target control problems has been discussed in [15]. In our work, for the time variant function  $M(\cdot)$ , we can also obtain that equivalence result.

We organize this paper as follows. In Section 2, we prove the time-varying bang-bang property (Theorem 1.1). In Section 3, we prove the equivalence theorem of optimal time and target control problems (Theorem 1.2).

## 2 Proof of Theorem 1.1

The following lemma is crucial in the proof of Theorem 1.1.

**Lemma 2.1.** *Under the assumption of Theorem 1.1, let  $\tau(\varepsilon)$  be defined as (1.2). Then there exists an optimal control  $u^* \in \mathcal{U}_M$ , such that the optimal time problem (1.2) is solvable, i.e.,*

$$y(T; \chi_{(\tau(\varepsilon), T)} u^*, y_0) \in \bar{B}(0, \varepsilon).$$

*Proof.* Since  $\mathcal{R}(y_0, T) \cap \bar{B}(0, \varepsilon) \neq \emptyset$ , there exist  $\tau_0 \in (0, T)$  and  $u \in \mathcal{U}_M$ , such that  $y(T; \chi_{(\tau_0, T)} u, y_0) \in \bar{B}(0, \varepsilon)$ . Let  $\{\tau_n\}_{n=1}^\infty$  be a monotonically increasing sequence such that  $\tau_n \rightarrow \tau(\varepsilon)$ . Then, for each  $n \in \mathbb{N}$ , there exists  $u_n \in \mathcal{U}_M$  such that

$$y(T; \chi_{(\tau_n, T)} u_n, y_0) \in \bar{B}(0, \varepsilon).$$

Set

$$\tilde{u}_n(t) = \begin{cases} 0, & \text{if } t \in (0, \tau_n], \\ u_n(t), & \text{if } t \in (\tau_n, T). \end{cases}$$

Since  $\|\tilde{u}_n\|_{L^\infty(0, T; L^2(\Omega))} \leq \|M\|_{L^\infty(0, T)}$ , there exist a subsequence of  $\{\tilde{u}_n\}$ , still denoted by itself, and  $u^* \in L^\infty(0, T; L^2(\Omega))$  such that

$$(2.1) \quad \tilde{u}_n \rightarrow u^* \text{ weakly}^* \text{ in } L^\infty(0, T; L^2(\Omega)).$$

Take  $t_0 \in (0, \tau(\varepsilon))$  to be the Lebesgue point of  $u^*$ , and  $\lambda \in (0, \frac{\tau(\varepsilon) - t_0}{2})$ . Then there exists  $N_0 \in \mathbb{N}$  such that  $t_0 + \lambda \leq \tau_n$  for all  $n \geq N_0$ . For any  $\zeta \in L^2(\Omega)$ , set

$$v(x, t) = \chi_{(t_0 - \lambda, t_0 + \lambda)}(t) \zeta(x).$$

Then  $\|v\|_{L^1(0, T; L^2(\Omega))} = 2\lambda \|\zeta\| < \infty$ , and

$$\begin{aligned} \int_{t_0 - \lambda}^{t_0 + \lambda} \langle u^*(t), \zeta \rangle dt &= \int_0^T \langle u^*(t), \chi_{(t_0 - \lambda, t_0 + \lambda)} \zeta \rangle dt \\ &= \int_0^T \langle u^*(t), v(t) \rangle dt \\ &= \lim_{n \rightarrow \infty} \int_0^T \langle \tilde{u}_n(t), v(t) \rangle dt \\ &= 0. \end{aligned}$$

Hence

$$\langle u^*(t_0), \zeta \rangle = \lim_{\lambda \rightarrow 0} \frac{1}{2\lambda} \int_{t_0 - \lambda}^{t_0 + \lambda} \langle u^*(t), \zeta \rangle dt = 0.$$

By the arbitrary of  $\zeta \in L^2(\Omega)$ , we get  $\tilde{u}^*(t_0) = 0$ . Since the Lebesgue measure of the set of  $u^*$ 's Lebesgue points in  $(0, \tau(\varepsilon))$  is equal to  $\tau(\varepsilon)$ , we have

$$(2.2) \quad u^*|_{(0, \tau(\varepsilon))} = 0.$$

On the other side, by (2.1), the solution  $y^*(\cdot; \chi_{(\tau(\varepsilon), T)} u^*, y_0)$  to

$$\begin{cases} \partial_t y^*(x, t) - \Delta y^*(x, t) = \chi_\omega \chi_{(\tau(\varepsilon), T)} u^*(x, t), & \text{in } \Omega \times (0, T), \\ y^*(x, t) = 0, & \text{on } \partial\Omega \times (0, T), \\ y^*(x, 0) = y_0(x), & \text{in } \Omega, \end{cases}$$

satisfies

$$(2.3) \quad \begin{aligned} y_n &\rightarrow y^* \text{ weakly in } L^2(0, T; H_0^1(\Omega)) \cap H^1(0, T; L^2(\Omega)), \\ &\text{strongly in } C([\delta, T]; L^2(\Omega)) \text{ as } n \rightarrow \infty, \end{aligned}$$

for any  $0 < \delta < T$ . Here  $y_n \equiv y_n(\cdot; \chi_{(\tau_n, T)} u_n, y_0)$  is the solution to the system

$$\begin{cases} \partial_t y_n(x, t) - \Delta y_n(x, t) = \chi_\omega \chi_{(\tau_n, T)} \tilde{u}_n(x, t), & \text{in } \Omega \times (0, T), \\ y_n(x, t) = 0, & \text{on } \partial\Omega \times (0, T), \\ y_n(x, 0) = y_0(x), & \text{in } \Omega. \end{cases}$$

Since  $y_n(T; \chi_{(\tau_n, T)} u_n, y_0) \in \bar{B}(0, \varepsilon)$ , we get  $y^*(T; \chi_{(\tau(\varepsilon), T)} u^*, y_0) \in \bar{B}(0, \varepsilon)$  by (2.3), which implies the optimal time  $\tau(\varepsilon)$  is attainable and the optimal control  $u^*$  exists.

We claim that  $\|u^*(t)\| \leq M(t)$  for a.e.  $t \in (\tau(\varepsilon), T)$ .

Indeed, if there exists  $\varepsilon_0 > 0$  and  $E_0 \subset [\tau(\varepsilon), T]$  with  $|E_0| > 0$  such that

$$\|u^*(t)\| > M(t) + \varepsilon_0, \quad \forall t \in E_0,$$

where  $|E_0|$  represents the Lebesgue measure of  $E_0$ . Define

$$\zeta(x, t) = \chi_{E_0} \frac{u^*(t)}{\|u^*(t)\|}.$$

It is obviously that  $\zeta(x, t)$  is well-defined since  $\|u^*(\cdot)\| > M(\cdot) + \varepsilon_0 > 0$  in  $E_0$ , and

$$\|\zeta\|_{L^1(\tau(\varepsilon), T; L^2(\Omega))} = \int_{\tau(\varepsilon)}^T \|\zeta(t)\| dt = \int_{\tau(\varepsilon)}^T \left\| \chi_{E_0} \frac{u^*(t)}{\|u^*(t)\|} \right\| dt = |E_0| < \infty,$$

i.e.,  $\zeta \in L^1(\tau(\varepsilon), T; L^2(\Omega))$ .

Since  $\tilde{u}_n \rightarrow \tilde{u}^*$  weakly\* in  $L^\infty(0, T; L^2(\Omega))$ , we get  $u_n \rightarrow u^*$  weakly\* in  $L^\infty(\tau(\varepsilon), T; L^2(\Omega))$ . For any  $\epsilon \in (0, \frac{|E_0|\varepsilon_0}{2})$ , there exists  $N_0 > 0$ , such that, for any  $n \geq N_0$ ,

$$(2.4) \quad \left| \int_{\tau(\varepsilon)}^T \langle u_n - u^*, \zeta \rangle dt \right| < \epsilon.$$

Noting

$$\|\zeta(t)\| = \left\| \chi_{E_0} \frac{u^*(t)}{\|u^*(t)\|} \right\| = \chi_{E_0},$$

we obtain

$$\begin{aligned} \left| \int_{\tau(\varepsilon)}^T \langle u_n - u^*, \zeta \rangle dt \right| &= \left| \int_{\tau(\varepsilon)}^T \langle u_n, \zeta \rangle dt - \int_{\tau(\varepsilon)}^T \langle u^*, \zeta \rangle dt \right| \\ &\geq \left| \int_{\tau(\varepsilon)}^T \langle u^*, \zeta \rangle dt \right| - \left| \int_{\tau(\varepsilon)}^T \langle u_n, \zeta \rangle dt \right| \\ &\geq \int_{\tau(\varepsilon)}^T \left\langle u^*, \chi_{E_0} \frac{u^*(t)}{\|u^*(t)\|} \right\rangle dt - \int_{\tau(\varepsilon)}^T \|u_n(t)\| \|\zeta(t)\| dt \\ &= \int_{E_0} \|u^*(t)\| dt - \int_{E_0} \|u_n(t)\| dt \\ &\geq \int_{E_0} (M(t) + \varepsilon_0) dt - \int_{E_0} M(t) dt \\ &= |E_0| \varepsilon_0. \end{aligned}$$

This contradicts (2.4). That proves our claim, and completes the proof. ■

Now we can prove Theorem 1.1.

*Proof of Theorem 1.1.* The proof is long, we separate it to two steps.

**Step 1.** By Lemma 2.1, one knows that  $\mathcal{R}(y_0, \tau(\varepsilon), T) \cap \bar{B}(0, \varepsilon) \neq \emptyset$ . We now show that  $\mathcal{R}(y_0, \tau(\varepsilon), T) \cap \bar{B}(0, \varepsilon)$  has only one point.

Otherwise, there exists at least two different  $u_1^*, u_2^* \in \mathcal{U}_M$  such that

$$y_1 \equiv y(T; \chi_{(\tau(\varepsilon), T)} u_1^*, y_0), y_2 \equiv y(T; \chi_{(\tau(\varepsilon), T)} u_2^*, y_0) \in \bar{B}(0, \varepsilon)$$

and

$$y_1 \neq y_2.$$

Note that  $\hat{u} = \frac{u_1^*}{2} + \frac{u_2^*}{2} \in \mathcal{U}_M$  and  $\hat{y}(\cdot) = \frac{1}{2}y_1(\cdot; \chi_{(\tau(\varepsilon), T)} u_1^*, y_0) + \frac{1}{2}y_2(\cdot; \chi_{(\tau(\varepsilon), T)} u_2^*, y_0)$  is the solution to the system

$$\begin{cases} \partial_t \hat{y}(x, t) - \Delta \hat{y}(x, t) = \chi_\omega \chi_{(\tau(\varepsilon), T)} \hat{u}(x, t), & \text{in } \Omega \times (0, T), \\ \hat{y}(x, t) = 0, & \text{on } \partial\Omega \times (0, T), \\ \hat{y}(x, 0) = y_0(x), & \text{in } \Omega. \end{cases}$$

One can easily get  $\hat{y}(T) = \frac{y_1}{2} + \frac{y_2}{2} \in \bar{B}(0, \varepsilon)$ . Since  $\bar{B}(0, \varepsilon) \subset L^2(\Omega)$  is strictly convex, we get  $\hat{y}(T)$  is an inner point of  $\bar{B}(0, \varepsilon)$ . Hence there exists  $\gamma > 0$  such that  $B(\hat{y}(T), \gamma) \subset B(0, \varepsilon)$ . Let  $\tilde{y}$  be the solution to the following system

$$\begin{cases} \partial_t \tilde{y}(x, t) - \Delta \tilde{y}(x, t) = \chi_\omega \chi_{(\tau(\varepsilon) + \xi, T)} \hat{u}(x, t), & \text{in } \Omega \times (0, T), \\ \tilde{y}(x, t) = 0, & \text{on } \partial\Omega \times (0, T), \\ \tilde{y}(x, 0) = y_0(x), & \text{in } \Omega. \end{cases}$$

Then

$$h_\xi \equiv \hat{y}(T) - \tilde{y}(T) = \int_{\tau(\varepsilon)}^{\tau(\varepsilon) + \xi} e^{\Delta(T-\sigma)} \chi_\omega \hat{u}(\sigma) d\sigma.$$

Choosing  $\xi > 0$  small enough such that  $\|h_\xi\| < \gamma$ , one has  $\tilde{y}(T) \in \bar{B}(0, \varepsilon)$ . That implies  $\tau(\varepsilon) \geq \tau(\varepsilon) + \xi$ , which is impossible. That completes the proof of Step 1.

**Step 2.** The optimal time control  $u^*$  has the time-varying bang-bang property.

Since  $\mathcal{R}(y_0, \tau(\varepsilon), T) \cap \bar{B}(0, \varepsilon)$  has only one point (denoted by  $y^* = y(T; u^*, y_0)$ ), and  $\mathcal{R}(y_0, \tau(\varepsilon), T)$  and  $\bar{B}(0, \varepsilon)$  are two convex sets, by hyperplane separation theorem, there exists  $\eta^* \in L^2(\Omega)$  such that

$$(2.5) \quad \sup_{y \in \mathcal{R}(y_0, \tau(\varepsilon), T)} \langle y, \eta^* \rangle \leq \inf_{z \in \bar{B}(0, \varepsilon)} \langle z, \eta^* \rangle \leq \langle y^*, \eta^* \rangle.$$

Notice that the element in  $\mathcal{R}(y_0, \tau(\varepsilon), T)$  can be written by

$$y(T; \chi_{(\tau(\varepsilon), T)} u, y_0) = e^{\Delta T} y_0 + \int_0^T e^{\Delta(T-\sigma)} \chi_\omega \chi_{(\tau(\varepsilon), T)} u(\sigma) d\sigma.$$

Then by (2.5), one can get

$$\sup_{\bar{u} \in \mathcal{U}_1} \int_{\tau(\varepsilon)}^T \left\langle e^{\Delta(T-\sigma)} \chi_\omega M(\sigma) \bar{u}(\sigma), \eta^* \right\rangle d\sigma \leq \int_{\tau(\varepsilon)}^T \left\langle e^{\Delta(T-\sigma)} \chi_\omega M(\sigma) \bar{u}^*(\sigma), \eta^* \right\rangle d\sigma,$$

i.e.,

$$(2.6) \quad \sup_{\bar{u} \in \mathcal{U}_1} \int_{\tau(\varepsilon)}^T \left\langle \bar{u}(\sigma), e^{\Delta(T-\sigma)} \chi_\omega M(\sigma) \eta^* \right\rangle d\sigma \leq \int_{\tau(\varepsilon)}^T \left\langle \bar{u}^*(\sigma), e^{\Delta(T-\sigma)} \chi_\omega M(\sigma) \eta^* \right\rangle d\sigma.$$

Here

$$\bar{u}^* \in \mathcal{U}_1 \equiv \left\{ \bar{u} \in L^\infty(\tau(\varepsilon), T; L^2(\Omega)) \mid \|\bar{u}(t)\|_{L^2(\Omega)} \leq 1 \text{ for a.e. } t \in [\tau(\varepsilon), T] \right\},$$

and

$$(2.7) \quad u^*(t) = M(t) \bar{u}^*(t) \text{ for all } t \in [\tau(\varepsilon), T].$$

Let  $E_0$  be the Lebesgue points of  $\bar{u}^*(\cdot)$  in  $[\tau(\varepsilon), T]$ . For given  $t_0 \in E_0$ , choosing

$$\bar{u}(t) = \begin{cases} \bar{u}^*(t), & \text{for } t \in (\tau(\varepsilon), T) \setminus (t_0 - \lambda, t_0 + \lambda), \\ \zeta, & \text{for } t \in (t_0 - \lambda, t_0 + \lambda) \subset (\tau(\varepsilon), T), \end{cases}$$

where  $\zeta \in L^2(\Omega)$  with  $\|\zeta\| \leq 1$ , and  $\lambda \in (0, \min\{t_0 - \tau(\varepsilon), T - t_0\})$ . Setting  $A = (\tau(\varepsilon), T) \setminus (t_0 - \lambda, t_0 + \lambda)$ , by (2.6) we have

$$\begin{aligned} & \int_A \langle \bar{u}^*(t), e^{\Delta(T-\sigma)} M(\sigma) \eta^* \rangle d\sigma + \int_{t_0-\lambda}^{t_0+\lambda} \langle \zeta, e^{\Delta(T-\sigma)} \chi_\omega M(\sigma) \eta^* \rangle d\sigma \\ & \leq \int_A \langle \bar{u}^*(t), e^{\Delta(T-\sigma)} M(\sigma) \eta^* \rangle d\sigma + \int_{t_0-\lambda}^{t_0+\lambda} \langle \bar{u}^*(\sigma), e^{\Delta(T-\sigma)} \chi_\omega M(\sigma) \eta^* \rangle d\sigma, \end{aligned}$$

i.e.,

$$\int_{t_0-\lambda}^{t_0+\lambda} \langle \zeta, e^{\Delta(T-\sigma)} \chi_\omega M(\sigma) \eta^* \rangle d\sigma \leq \int_{t_0-\lambda}^{t_0+\lambda} \langle \bar{u}^*(\sigma), e^{\Delta(T-\sigma)} \chi_\omega M(\sigma) \eta^* \rangle d\sigma.$$

Hence

$$\begin{aligned} & \langle \zeta, e^{\Delta(T-t_0)} \chi_\omega M(t_0) \eta^* \rangle \\ & = \lim_{\lambda \rightarrow 0} \frac{1}{2\lambda} \int_{t_0-\lambda}^{t_0+\lambda} \langle \zeta, e^{\Delta(T-\sigma)} \chi_\omega M(\sigma) \eta^* \rangle d\sigma \\ & \leq \lim_{\lambda \rightarrow 0} \frac{1}{2\lambda} \int_{t_0-\lambda}^{t_0+\lambda} \langle \bar{u}^*(\sigma), e^{\Delta(T-\sigma)} \chi_\omega M(\sigma) \eta^* \rangle d\sigma \\ & = \langle \bar{u}^*(t_0), e^{\Delta(T-t_0)} \chi_\omega M(t_0) \eta^* \rangle. \end{aligned}$$

By the arbitrary of  $\zeta$ , we get

$$\sup_{\|\zeta\| \leq 1} \left\langle \zeta, e^{\Delta(T-t_0)} \chi_\omega M(t_0) \eta^* \right\rangle \leq \left\langle \bar{u}^*(t_0), e^{\Delta(T-t_0)} \chi_\omega M(t_0) \eta^* \right\rangle,$$

i.e.,

$$\begin{aligned} \left\| e^{\Delta(T-t_0)} \chi_\omega M(t_0) \eta^* \right\|_{L^2(\Omega)} & = \sup_{\|\zeta\| \leq 1} \left\langle \zeta, e^{\Delta(T-t_0)} \chi_\omega M(t_0) \eta^* \right\rangle \\ & \leq \left\langle \bar{u}^*(t_0), e^{\Delta(T-t_0)} \chi_\omega M(t_0) \eta^* \right\rangle \\ & \leq \|\bar{u}^*(t_0)\|_{L^2(\Omega)} \left\| e^{\Delta(T-t_0)} \chi_\omega M(t_0) \eta^* \right\|_{L^2(\Omega)}. \end{aligned}$$

This implies that

$$(2.8) \quad \|\bar{u}^*(t_0)\|_{L^2(\Omega)} \geq 1.$$

By  $\bar{u}^* \in \mathcal{U}_1$  we get  $\|\bar{u}^*(t_0)\| = 1$ . (2.8), together with (2.7) and  $|E_0| = \hat{\tau}(\varepsilon)$ , yields

$$\|u^*(t)\|_{L^2(\Omega)} = M(t) \text{ for a.e. } t \in [0, \hat{\tau}(\varepsilon)].$$

From above, we get the time optimal control  $u^*$  satisfies the time-varying bang-bang property. That completes the proof.  $\blacksquare$

### 3 Proof of Theorem 1.2

In order to show Theorem 1.2, we need a lemma in the following:

**Lemma 3.1.** *Let  $\varepsilon(\tau)$  be defined as (1.4). Then there exists  $u^* \in \mathcal{U}_M$  such that*

$$y(T; \chi_{(\tau, T)} u^*, y_0) \in \bar{B}(0, \varepsilon(\tau)).$$

*Proof.* Let  $\{u_n\}_{n=1}^\infty \subset \mathcal{U}_M$  be the minimal sequence of (1.4), i.e.,

$$\|y(T; \chi_{(\tau, T)} u_n, y_0)\| \rightarrow \varepsilon(\tau), \text{ as } n \rightarrow \infty.$$

Here  $y(\cdot; \chi_{(\tau, T)} u_n, y_0)$  is the solution to the system

$$\begin{cases} \partial_t y(x, t) - \Delta y(x, t) = \chi_\omega \chi_{(\tau, T)} u_n(x, t), & \text{in } \Omega \times (0, T), \\ y(x, t) = 0, & \text{on } \partial\Omega \times (0, T), \\ y(x, 0) = y_0(x), & \text{in } \Omega. \end{cases}$$

Since

$$\|u_n(t)\| \leq M(t) \leq \|M\|_{L^\infty(0, T)} \text{ for a.e. } t \in (\tau, T),$$

there exist a subsequence of  $\{u_n\}_{n=1}^\infty$ , still denoted by itself, and  $u^* \in L^\infty(\tau, T; L^2(\Omega))$  such that

$$u_n \rightarrow u^* \text{ weakly}^* \text{ in } L^\infty(\tau, T; L^2(\Omega)).$$

Therefore, we have

$$\begin{aligned} y(\cdot; \chi_{(\tau, T)} u_n, y_0) &\rightarrow y(\cdot; \chi_{(\tau, T)} u^*, y_0) \text{ weakly in } L^2(0, T; H_0^1(\Omega)) \cap H^1(0, T; L^2(\Omega)), \\ &\text{strongly in } C([\delta, T]; L^2(\Omega)) \text{ as } n \rightarrow \infty, \end{aligned}$$

for any  $\delta \in (0, T]$ . Here  $y(\cdot; \chi_{(\tau, T)} u^*, y_0)$  is the solution to the system

$$\begin{cases} \partial_t y(x, t) - \Delta y(x, t) = \chi_\omega \chi_{(\tau, T)} u^*(x, t), & \text{in } \Omega \times (0, T), \\ y(x, t) = 0, & \text{on } \partial\Omega \times (0, T), \\ y(x, 0) = y_0(x), & \text{in } \Omega. \end{cases}$$

Hence,

$$\|y(T; \chi_{(\tau, T)} u^*, y_0)\| = \varepsilon(\tau).$$



Now, we shall show that  $u^* \in \mathcal{U}_M$ .

By contradiction. We assume there exist  $\epsilon_0 > 0$  and  $E_0 \subset (\tau, T)$  with  $|E_0| > 0$  such that

$$u^*(t) \geq M(t) + \epsilon_0 \text{ for all } t \in E_0.$$

Then

$$(3.1) \quad \int_{E_0} \|u^*(t)\| dt \geq \int_{E_0} (M(t) + \epsilon_0) dt = \int_{E_0} M(t) dt + \epsilon_0 |E_0|.$$

Taking  $\zeta(x, t) = \chi_{E_0} \frac{u^*(t)}{\|u^*(t)\|}$ , by the assumption of  $u^*$ , we get

$$\int_0^T \|\zeta(x, t)\| dt = \int_0^T \left\| \chi_{E_0} \frac{u^*(t)}{\|u^*(t)\|} \right\| dt = |E_0|,$$

i.e.,  $\zeta \in L^1(0, T; L^2(\Omega))$ . Hence, by  $u_n \rightarrow u^*$  weakly\* in  $L^\infty(\tau, T; L^2(\Omega))$ , one has

$$\int_\tau^T \left\langle u_n(t), \chi_{E_0} \frac{u^*(t)}{\|u^*(t)\|} \right\rangle dt = \int_\tau^T \langle u_n(t), \zeta \rangle dt \rightarrow \int_\tau^T \langle u^*(t), \zeta(t) \rangle dt = \int_{E_0} \|u^*(t)\| dt,$$

as  $n \rightarrow \infty$ . On the other hand, since  $u_n \rightarrow u^*$  weakly\* in  $L^\infty(\tau, T; L^2(\Omega))$  and  $E_0 \subset (\tau, T)$ , we get  $u_n \rightarrow u^*$  weakly\* in  $L^\infty(E_0; L^2(\Omega))$ . Therefore, by  $\|\zeta(t)\| = \left\| \chi_{E_0} \frac{u^*(t)}{\|u^*(t)\|} \right\| = 1$  and  $\|u_n(t)\| \leq M(t)$  for a.e.  $t \in (\tau, T)$ , one gets

$$\begin{aligned} \int_{E_0} \|u^*(t)\| dt &= \lim_{n \rightarrow \infty} \int_{E_0} \langle u_n(t), \zeta(t) \rangle dt \\ &\leq \lim_{n \rightarrow \infty} \int_{E_0} \|u_n(t)\| \|\zeta(t)\| dt \\ &= \lim_{n \rightarrow \infty} \int_{E_0} \|u_n(t)\| dt \\ &\leq \int_{E_0} M(t) dt, \end{aligned}$$

which is contradict with (3.1). That completes the proof. ■

Now, we are in the position to prove Theorem 1.2.

*Proof of Theorem 1.2.* We carry out the proof by three steps as follows.

**Step 1.** We shall show that  $\tau \mapsto \varepsilon(\tau)$  is strictly monotonically increasing.

Let  $0 \leq \tau_1 < \tau_2 < T$ . For  $\tau_2$ , by Theorem 1.1, there exists a unique  $u_2 \in \mathcal{U}_M$  such that

$$\|y(T; \chi_{(\tau_2, T)} u_2, y_0)\| = \varepsilon(\tau_2).$$

Now, take

$$u_1(t) = \begin{cases} u_2(t), & \text{if } t \in (\tau_2, T), \\ 0, & \text{if } t \in [0, \tau_2]. \end{cases}$$

Then

$$\|y(T; \chi_{(\tau_1, T)} u_1, y_0)\| = \|y(T; \chi_{(\tau_2, T)} u_2, y_0)\| \in \bar{B}(0, \varepsilon(\tau_2)).$$

By the definition of  $\varepsilon(\tau)$  we get

$$\varepsilon(\tau_1) \leq \varepsilon(\tau_2).$$

In other words,  $\tau \mapsto \varepsilon(\tau)$  is a monotonically increasing function.

Now, we show that  $\tau \mapsto \varepsilon(\tau)$  is strictly monotonically increasing. If not, suppose that  $\varepsilon(\tau_1) = \varepsilon(\tau_2)$ . Then, there exist  $u_1, u_2 \in \mathcal{U}_M$ , such that

$$\|y(T; \chi_{(\tau_1, T)} u_1, y_0)\| = \varepsilon(\tau_1) = \varepsilon(\tau_2) = \|y(T; \chi_{(\tau_2, T)} u_2, y_0)\|.$$

Taking

$$\hat{u}_2(t) = \begin{cases} u_2(t), & \text{if } t \in (\tau_2, T), \\ 0, & \text{if } t \in [0, \tau_2]. \end{cases}$$

one can easily check that

$$\|y(T; \chi_{(\tau_1, T)} u_1, y_0)\| = \varepsilon(\tau_1) = \|y(T; \chi_{(\tau_1, T)} \hat{u}_2, y_0)\|.$$

By Theorem 1.1, the optimal control is unique. Hence, we get

$$u_1(t) = \hat{u}_2(t) \text{ for a.e. } t \in (0, T).$$

By the bang-bang property of minimal control, we have

$$\|u_1(t)\| = \|\hat{u}_2(t)\| = M(t) \text{ for a.e. } t \in (0, T).$$

That is impossible, since  $\hat{u}_2(t) = 0$  for a.e.  $t \in (\tau_1, \tau_2)$  and  $M(t) > 0$  for a.e.  $t \in (0, T)$ . Therefore,  $\tau \mapsto \varepsilon(\tau)$  is a strictly monotonically increasing function.

**Step 2.** We shall show that  $\tau \mapsto \varepsilon(\tau)$  is continuous.

For  $\tau, \hat{\tau} \in [0, T]$  with  $\tau < \hat{\tau}$ , and for each  $u \in \mathcal{U}_M$ , the solutions to (1.1) are

$$y(T; \chi_{(\tau, T)} u, y_0) = e^{\Delta T} y_0 + \int_0^T e^{\Delta(T-\sigma)} \chi_\omega \chi_{(\tau, T)} u(\sigma) d\sigma$$

and

$$y(T; \chi_{(\hat{\tau}, T)} u, y_0) = e^{\Delta T} y_0 + \int_0^T e^{\Delta(T-\sigma)} \chi_\omega \chi_{(\hat{\tau}, T)} u(\sigma) d\sigma,$$

respectively. Then

$$\begin{aligned} & \|y(T; \chi_{(\tau, T)} u, y_0) - y(T; \chi_{(\hat{\tau}, T)} u, y_0)\| \\ (3.2) \quad &= \left\| \int_0^T e^{\Delta(T-\sigma)} \chi_\omega \chi_{(\tau, T)} u(\sigma) d\sigma - \int_0^T e^{\Delta(T-\sigma)} \chi_\omega \chi_{(\hat{\tau}, T)} u(\sigma) d\sigma \right\| \\ &= \left\| \int_\tau^{\hat{\tau}} e^{\Delta(T-\sigma)} \chi_\omega u(\sigma) d\sigma \right\| \\ &\leq C|\hat{\tau} - \tau|, \end{aligned}$$

where  $C$  is a constant independent of  $\tau, \hat{\tau}$ . Since  $u \in \mathcal{U}_M$  (i.e.,  $\|u(t)\| \leq M(t) \leq M_T$  for a.e.  $t \in [0, T]$ ), by (3.2), we get  $\text{dist}(\mathcal{R}(y_0, \tau, T), \mathcal{R}(y_0, \hat{\tau}, T)) < C|\tau - \hat{\tau}|$ . Here  $\text{dist}$  is the distance of two reachable sets  $\mathcal{R}(y_0, \tau, T)$  and  $\mathcal{R}(y_0, \hat{\tau}, T)$ . Hence  $\tau \mapsto \varepsilon(\tau)$  is a continuous function.

**Step 3.** We shall prove (1.5).

(1) We show that  $\varepsilon = \varepsilon(\tau(\varepsilon))$  for  $\varepsilon \in [\varepsilon(0), \varepsilon_T]$ .

Let  $\varepsilon \in [\varepsilon(0), \varepsilon_T]$ . By Step 1 in the proof of Theorem 1.1, there exist  $\tau(\varepsilon) \in [0, T]$  and  $u \in \mathcal{U}_M$  such that

$$(3.3) \quad \|y(T; \chi_{(\tau(\varepsilon), T)} u, y_0)\| = \varepsilon.$$

For such  $\tau(\varepsilon) \in [0, T]$ , denote  $\tau^* = \tau(\varepsilon)$ . We consider the following problem

$$\varepsilon(\tau^*) = \inf_{u \in \mathcal{U}_M} \|y(T; \chi_{(\tau^*, T)} u, y_0)\|.$$

By (3.3), we can obtain

$$(3.4) \quad \varepsilon(\tau^*) \leq \varepsilon.$$

By lemma 3.1, there exists a control  $u^* \in \mathcal{U}_M$  such that

$$\|y(T; \chi_{(\tau^*, T)} u^*, y_0)\| = \varepsilon(\tau^*).$$

Now, taking

$$\tilde{u}(t) = \begin{cases} u^*(t), & \text{if } t \in (\tau^*, T), \\ 0, & \text{if } t \in [0, \tau^*], \end{cases}$$

we have

$$\|y(T; \chi_{(\tau^*, T)} \tilde{u}, y_0)\| = \varepsilon(\tau^*).$$

By the definition of  $\tau(\varepsilon)$  and (3.3) we get

$$\varepsilon \leq \varepsilon(\tau^*).$$

which, together with (3.4), yields

$$\varepsilon = \varepsilon(\tau(\varepsilon)).$$

(2) We show that  $\tau = \tau(\varepsilon(\tau))$  for  $\tau \in [0, T]$ .

Let  $\tau \in [0, T]$ . By Lemma 3.1, there exists  $u \in \mathcal{U}_M$  such that

$$(3.5) \quad y(T; \chi_{(\tau, T)} u, y_0) \in \bar{B}(0, \varepsilon(\tau)).$$

For the given  $\varepsilon(\tau) \in [\varepsilon(0), \varepsilon_T]$ , denote  $\varepsilon^* = \varepsilon(\tau)$ . Consider the following problem

$$\tau(\varepsilon^*) = \sup\{\tilde{\tau} \mid y(T; \chi_{(\tilde{\tau}, T)} u, y_0) \in \bar{B}(0, \varepsilon^*), u \in \mathcal{U}_M\}.$$

Then we have

$$(3.6) \quad \tau \leq \tau(\varepsilon^*).$$

By Lemma 2.1, there exists a control  $u^* \in \mathcal{U}_M$  such that

$$y(T; \chi_{(\tau(\varepsilon^*), T)} u^*, y_0) \in \bar{B}(0, \varepsilon^*).$$

Then by the definition of  $\varepsilon(\tau)$  and (3.5) we get

$$\tau(\varepsilon^*) \leq \tau,$$

which, together with (3.6), yields  $\tau = \tau(\varepsilon(\tau))$  for  $\tau \in [0, T]$ . That completes the proof. ■

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